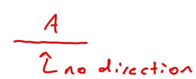
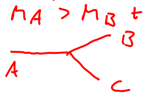
Monday, March 05, 2012
8:28 AM

The Feynman Rules (or how to calculate $\mathcal{M}$)

A very simple "toy" model:

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi_A\partial^\mu\phi_A - \tfrac{1}{2}\left(\tfrac{m_A c}{\hbar}\right)^2\phi_A^2 + \tfrac{1}{2}\partial_\mu\phi_B\partial^\mu\phi_B - \tfrac{1}{2}\left(\tfrac{m_B c}{\hbar}\right)^2\phi_B^2 + \tfrac{1}{2}\partial_\mu\phi_C\partial^\mu\phi_C - \tfrac{1}{2}\left(\tfrac{m_C c}{\hbar}\right)^2\phi_C^2 - g\,\phi_A\phi_B\phi_C$$

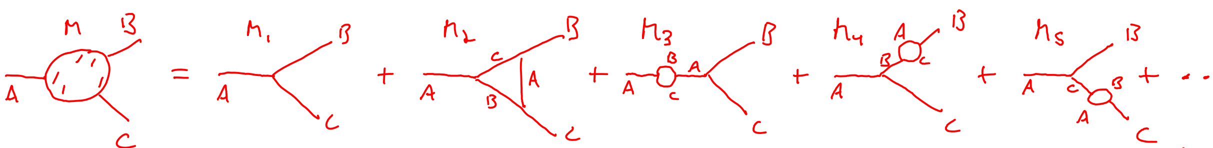
1) 3 real spin-0 particles A, B, C
2) They are their own antiparticles, e.g. $A = \overline{A}$ or 

3) $m_A > m_B + m_C$
4) 

basic interaction vertex

So far you know how to draw allowed diagrams given the rules. Now we want to "evaluate" them. Recall everything in this toy model is built from ⟨ w/ no directions, so you don't have to worry about correct "flow" e.g. 

Example: "Decay of A into B+C"



Tree-level — Vertex-correction — self-energy-corrections

1st order — 3rd order

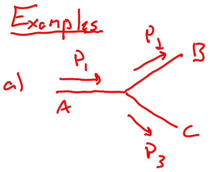
We evaluate each diagram to get $\mathcal{M}_i$ using these rules (will change somewhat for full SM):

1. Label all momenta p_i - external, q_i - internal w/ arrows next to lines. This lets us keep track of momentum flow (different from particle identity flow). For p_i the arrows must go forward in time, but for q_i it doesn't matter.
2. For each vertex write a factor of $-ig$ (g is the coupling strength ⟨).
3. For each internal propagator write a factor $i/(q_j^2 - m_j^2c^2)$ Note: $q_j^2 \neq m_j^2c^2$ since virtual
4. For each vertex conserve 4-momentum w/ $(2\pi)^4\delta^4(P_{tot,in} - P_{tot,out})$ where $P = \sum p_i, q_i$
5. Integrate everything you have written over all internal 4-momenta w/ "factors" $\int \frac{1}{(2\pi)^4} d^4q_j$
6. After this you will have an overall $(2\pi)^4\delta^4(p_{tot,in} - p_{tot,out})$. Erase this and multiply by i to get $\mathcal{M}$.

Tuesday, March 20, 2012
8:32 AM

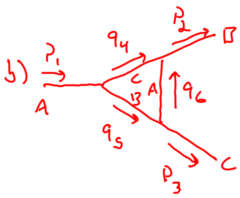
Examples

a)



$$-ig(2\pi)^4\delta^4(p_1-p_2-p_3) \Rightarrow M = g$$

b)



$$\iiint (-ig)^3 \frac{i}{q_4^2-m_C^2c^2}\frac{i}{q_5^2-m_B^2c^2}\frac{i}{q_6^2-m_A^2c^2}(2\pi)^4\delta^4(p_1-q_4-q_5)(2\pi)^4\delta^4(q_4+q_6-p_2)$$
$$\times(2\pi)^4\delta^4(q_5-q_6-p_3)\frac{d^4q_4}{(2\pi)^4}\frac{d^4q_5}{(2\pi)^4}\frac{d^4q_6}{(2\pi)^4}$$

Use $\delta^4(p_1-q_4-q_5)$ to do q_4 integral (replacing $q_4=p_1-q_5$ everywhere):

$$\iint (-ig)^3 \frac{i}{(p_1-q_5)^2-m_C^2c^2}\frac{i}{q_5^2-m_B^2c^2}\frac{i}{q_6^2-m_A^2c^2}(2\pi)^4\delta^4(p_1-q_5+q_6-p_2)$$
$$(2\pi)^4\delta^4(q_5-q_6-p_3)\frac{d^4q_5}{(2\pi)^4}\frac{d^4q_6}{(2\pi)^4}$$

Use $\delta^4(q_5-q_6-p_3)$ to do q_6 integral:

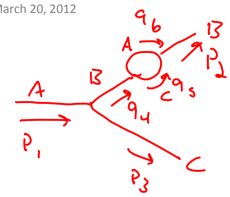
$$\int (-ig)^3 \frac{i}{(p_1-q_5)^2-m_C^2c^2}\frac{i}{q_5^2-m_B^2c^2}\frac{i}{(q_5-p_3)^2-m_A^2c^2}(2\pi)^4\delta^4(p_1-p_2-p_3)\frac{d^4q_5}{(2\pi)^4}$$

or

$$(2\pi)^4\delta^4(p_1-p_2-p_3)(-ig)^3 i^3\int\frac{1}{(p_1-q_5)^2-m_C^2c^2}\frac{1}{q_5^2-m_B^2c^2}\frac{1}{(q_5-p_3)^2-m_A^2c^2}\frac{d^4q_5}{(2\pi)^4}$$

Then:

$$M = i(-ig)^3 i^3\int\frac{1}{(p_1-q_5)^2-m_C^2c^2}\frac{1}{q_5^2-m_B^2c^2}\frac{1}{(q_5-p_3)^2-m_A^2c^2}\frac{d^4q_5}{(2\pi)^4}$$

c) 

$$\int\int\int (-ig)^3 \frac{i}{q_4^2 - m_B^2c^2}\,\frac{i}{q_5^2 - m_C^2c^2}\,\frac{i}{q_6^2 - m_A^2c^2}\,(2\pi)^4\delta^4(p_1 - q_4 - p_3)$$
$$\times (2\pi)^4\delta^4(q_4 - q_5 - q_6)$$
$$\times (2\pi)^4\delta^4(q_5 + q_6 - p_2)$$
$$\times \frac{d^4q_4}{(2\pi)^4}\,\frac{d^4q_5}{(2\pi)^4}\,\frac{d^4q_6}{(2\pi)^4}$$

Use $\delta(q_5 + q_6 - p_2)$ to do $\int d^4q_6$:

$$\int\int (-ig)^3 \frac{i}{q_4^2 - m_B^2c^2}\,\frac{i}{q_5^2 - m_C^2c^2}\,\frac{i}{(p_2 - q_5)^2 - m_A^2c^2}\,(2\pi)^4\delta^4(p_1 - q_4 - p_3)$$
$$\times (2\pi)^4\delta^4(q_4 - q_5 - p_2 + q_5)$$
$$\times \frac{d^4q_4}{(2\pi)^4}\,\frac{d^4q_5}{(2\pi)^4}$$

Use $\delta(q_4 - p_2)$ to do $\int d^4q_4$:

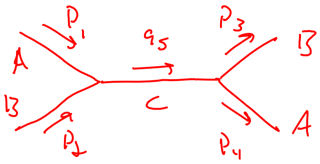
$$\int (-ig)^3 \frac{i}{p_2^2 - m_B^2c^2}\,\frac{i}{q_5^2 - m_C^2c^2}\,\frac{i}{(p_2 - q_5)^2 - m_A^2c^2}\,(2\pi)^4\delta^4(p_1 - p_2 - p_3)\,\frac{d^4q_5}{(2\pi)^4}$$

Then:

$$M = i(-ig)^3 i^3 \quad \frac{1}{p_2^2 - m_B^2c^2}\int \frac{1}{q_5^2 - m_C^2c^2}\,\frac{1}{(p_2 - q_5)^2 - m_A^2c^2}\,\frac{d^4q_5}{(2\pi)^4}$$

Tuesday, March 20, 2012
7:12 PM

As another example consider the scattering process $A+B \to A+B$ at lowest order:

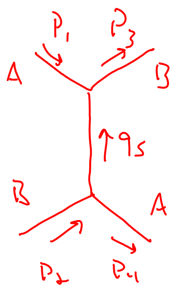


$$\int (-ig)^2 \frac{i}{q_5^2 - m_c^2 c^2} (2\pi)^4 \delta^4(p_1+p_2-q_5)\, \underbrace{(2\pi)^4 \delta^4(q_5-p_3-p_4)}_{\text{Use to set } q_5 = p_3+p_4} \frac{d^4 q_5}{(2\pi)^4}$$

$$(-ig)^2\, i \frac{1}{(p_3+p_4)^2 - m_c^2 c^2} (2\pi)^4 \delta^4(p_1+p_2-p_3-p_4)$$

Then:

$$\mathcal{M}_1 = \frac{g^2}{(p_3+p_4)^2 - m_c^2 c^2}$$



$$\int (-ig)^2 \frac{i}{q_5^2 - m_c^2 c^2} (2\pi)^4 \delta^4(p_1+q_5-p_3)\, (2\pi)^4 \delta^4(p_2-q_5-p_4) \frac{d^4 q_5}{(2\pi)^4}$$

$$(-ig)^2\, i \frac{1}{(p_2-p_4)^2 - m_c^2 c^2} (2\pi)^4 \delta^4(p_1+p_2-p_3-p_4)$$

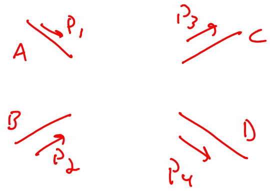
Then:

$$\mathcal{M}_2 = \frac{g^2}{(p_2-p_4)^2 - m_c^2 c^2}$$

So to leading order we have $\mathcal{M} = g^2 \left[\frac{1}{(p_3+p_4)^2 - m_c^2 c^2} + \frac{1}{(p_2-p_4)^2 - m_c^2 c^2} \right]$

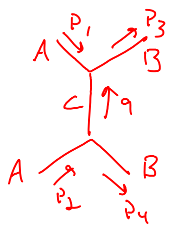
Tuesday, March 20, 2012
7:19 PM

When evaluating various Feynman diagrams (especially when you plan to add several contributions) it helps to start with all of the external labels:

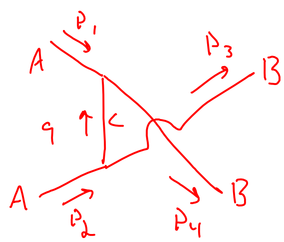


Then fill in the rest of the diagram. This way you can avoid mistakenly using a momentum p_i for different particles in different diagrams. It is important when adding this that a single external momentum label, e.g. p_1, always refer to the same particle.

What about



vs.



Are they different?
Should they both be included?

They are definitely different!

For example if $p_1 = 100$
$p_2 = 2$
$p_3 = 101$
$p_4 = 1$

Then for the first diagram $q = 1$

while for the second diagram $q = -99$

Both should be included as long as they are allowed by the rules of the theory!